Optimal Piecewise-based Mechanism for Collecting Bounded Numerical Data under Local Differential Privacy

Authors: **Ye Zheng**, Sumita Mishra, Yidan Hu

RIT | Rochester Institute of Technology

LDP Mechanisms

- Randomization algorithm $\mathcal{M}: \mathcal{D} \to \widetilde{\mathcal{D}}$

  - quantifiable privacy for data $x \in \mathcal{D}$

$$\forall x_1, x_2 \in \mathcal{D}, \forall y \in \widetilde{\mathcal{D}} \qquad \max \frac{\Pr[\mathcal{M}(x_1) = y]}{\Pr[\mathcal{M}(x_2) = y]} \leq e^{\varepsilon}$$

Distinguishability of x_1 and x_2 (sensitive data)
from y (randomized data)

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$x_1 \quad \to \quad \mathcal{M} \quad \to \quad y$ 👥 🛡️ 😈 Provable defense against data inference attack	

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Provable defense against
data inference attack

Data utility	by expected errors

dist of random y
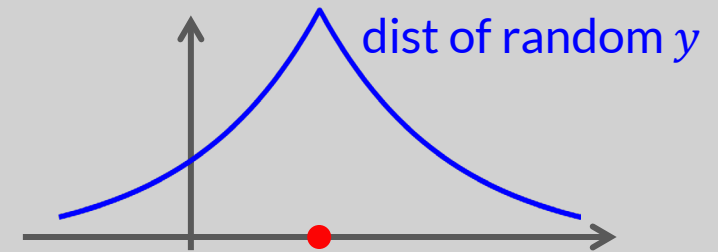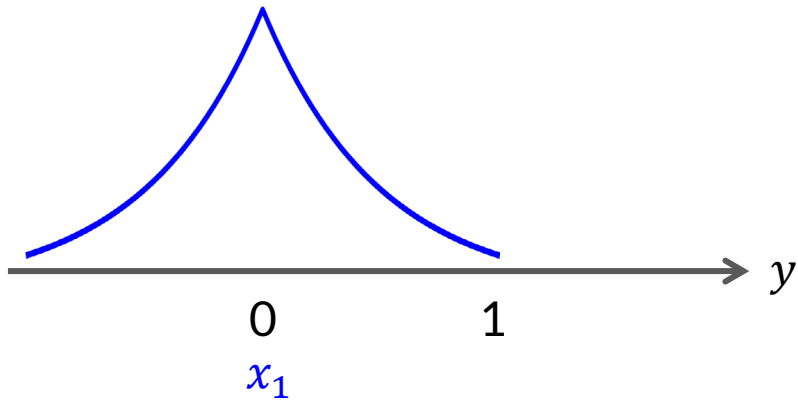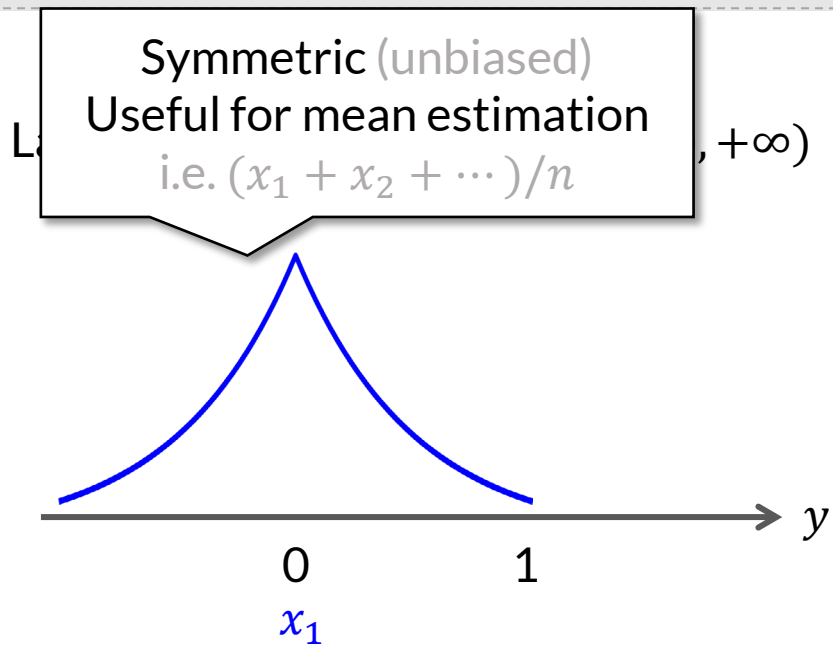
Sensitive $x_1 = 0.5$

LDP Mechanisms for Numerical Domain

Laplace mechanism: $[0,1] \rightarrow (-\infty, +\infty)$

LDP Mechanisms for Numerical Domain
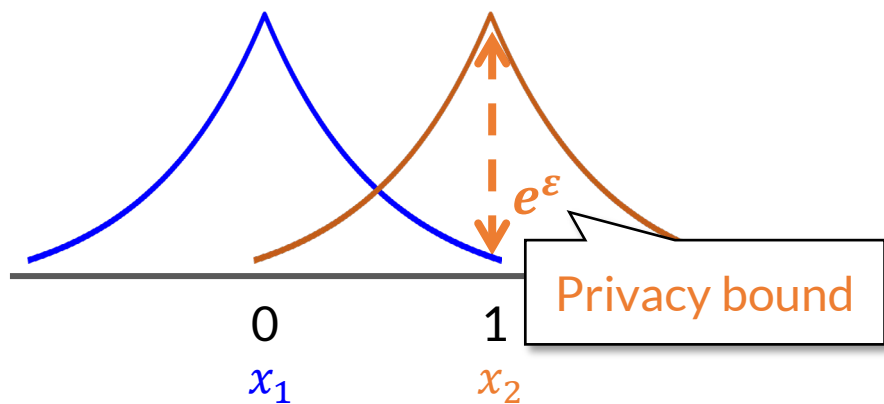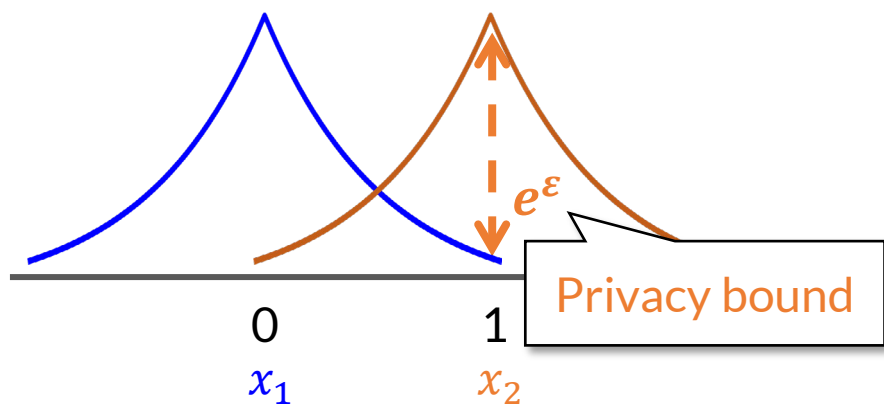

Symmetric (unbiased)
Useful for mean estimation
i.e. $(x_1 + x_2 + \cdots)/n$

L[,+∞)

x_1

LDP Mechanisms for Numerical Domain

Laplace mechanism: $[0,1] \rightarrow (-\infty, +\infty)$

e^{ε}

Privacy bound

0 1

x_1 x_2

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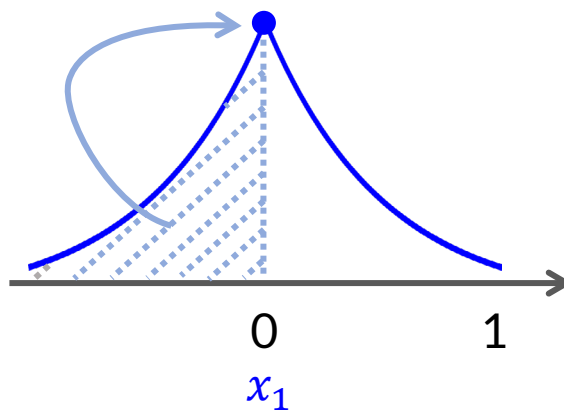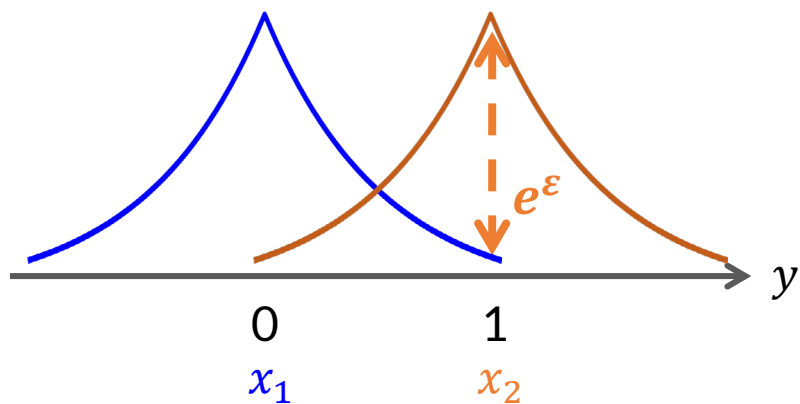# LDP Mechanisms for Numerical Domain

Laplace mechanism: $[0,1] \rightarrow (-\infty, +\infty)$

Laplace + truncation: $[0,1] \rightarrow [0,1]$

Smaller output
Useful for distribution estimation
i.e. $\text{dist}\{x_1, x_2, \cdots\}$

e^ε

y

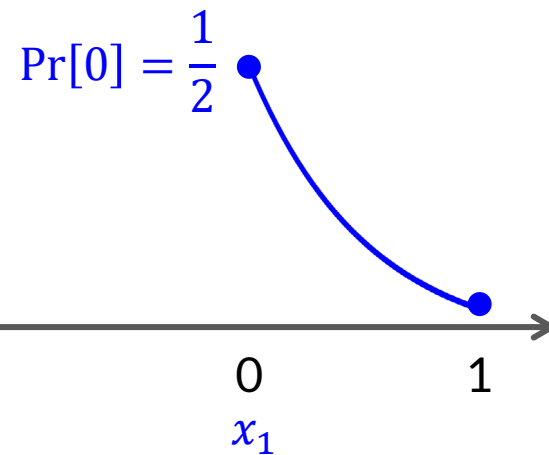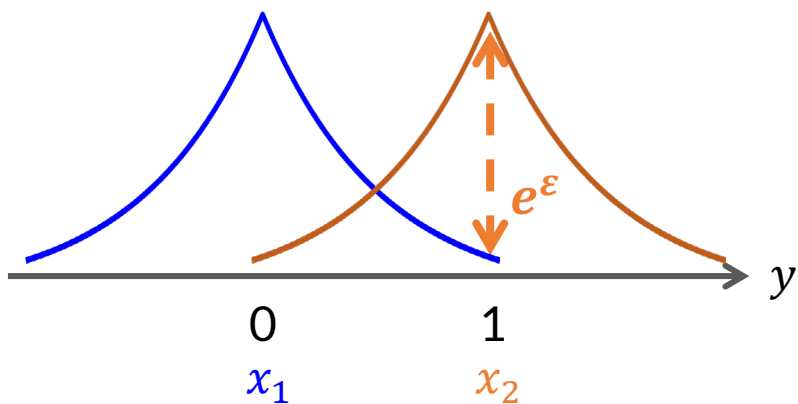0
x_1

1
x_2

$\Pr[0] = \dfrac{1}{2}$

0
x_1

1

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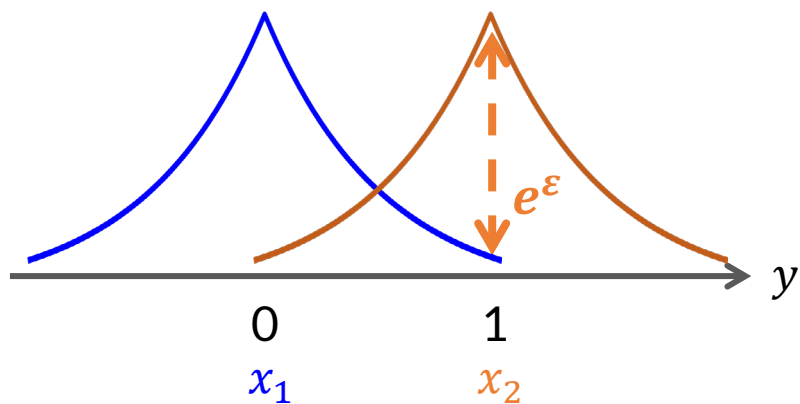
Staircase: $[0,1] \rightarrow (-\infty, +\infty)$

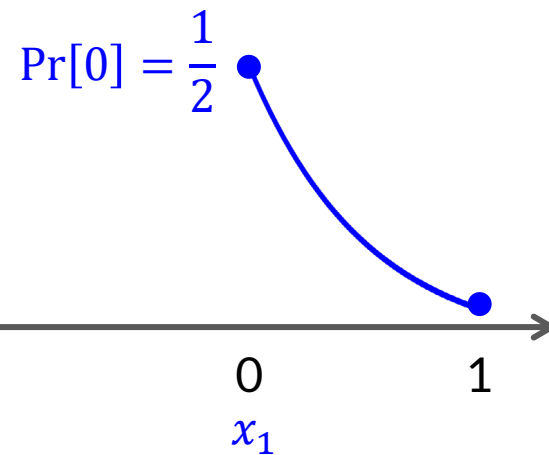Piecewise mechanism: $[0,1] \rightarrow [-C_\varepsilon, C_\varepsilon]$

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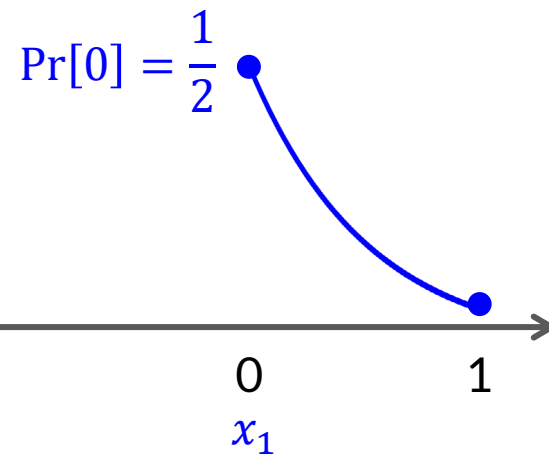
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e^{ε}

0 1 y

x_1 x_2

$\Pr[0] = \dfrac{1}{2}$

0 1

x_1

$\cdots$ $\cdots$

0 1

x_1

Piecewise mechanism: $[0,1] \rightarrow [-C_{\varepsilon}, C_{\varepsilon}]$

e^{ε}

$-C_{\varepsilon}$ 0 1 C_{ε}

x_1 x_2

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e^ε

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x_1

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$\dots$

$\dots$

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x_1

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$-C_\varepsilon$

0

1

C_ε

x_1
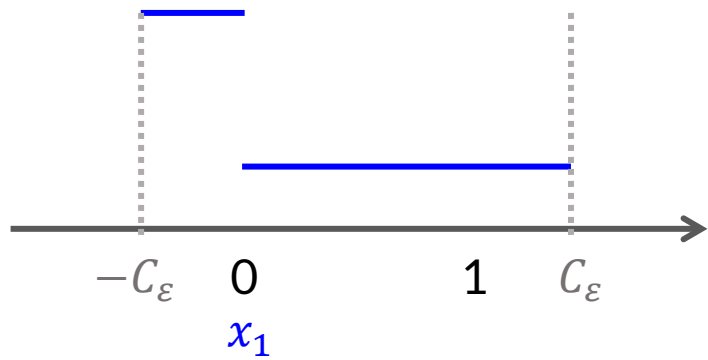
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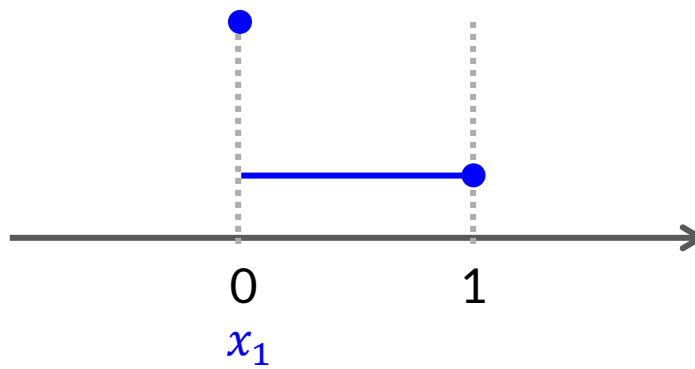
0

1

x_1

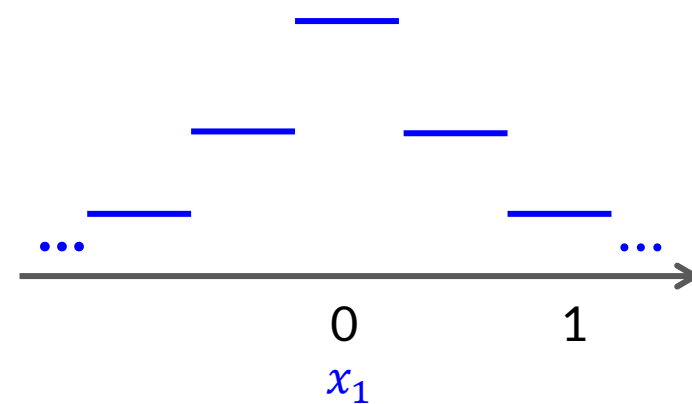$\bullet\bullet\bullet$

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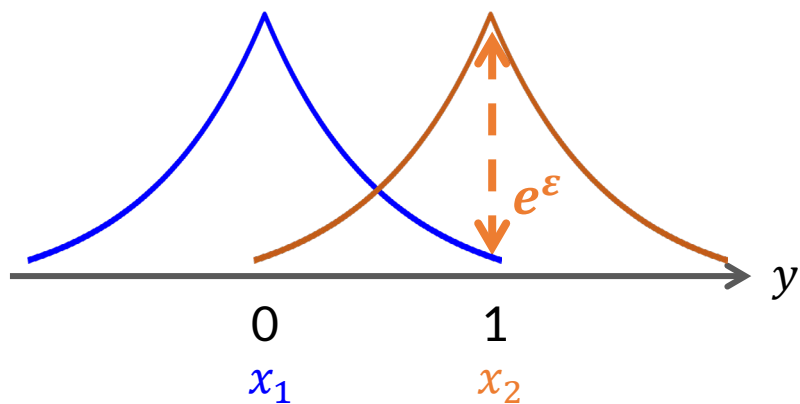e^{ε}

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Privacy: LDP with the **same** ε 　　　　　 **Utility:** **Different errors**

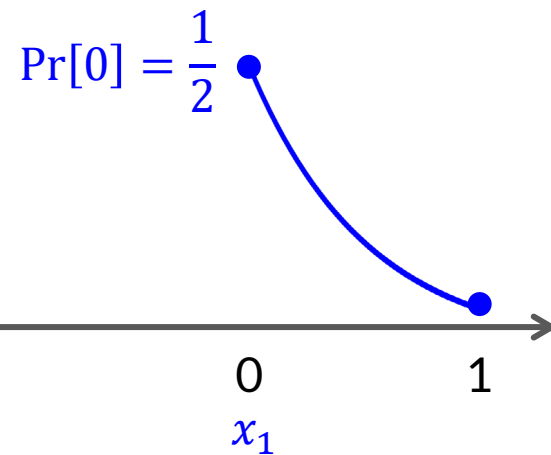$-C_{\varepsilon}$ 　 0 　　 1 　 C_{ε}
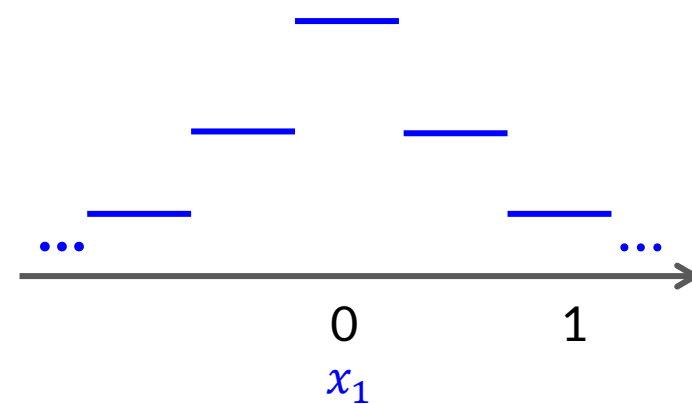
x_1

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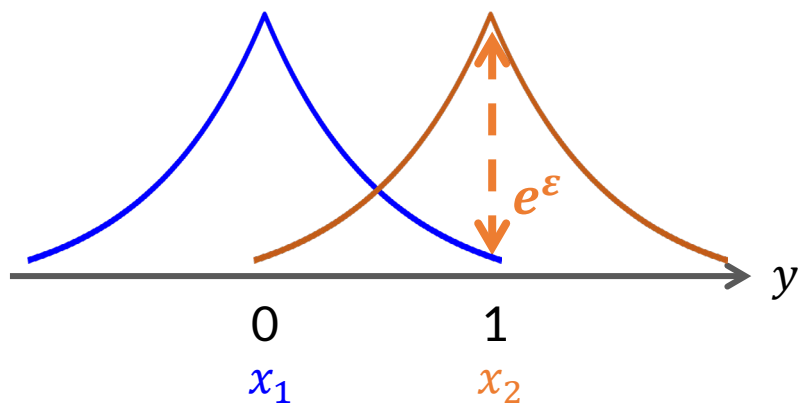e^{ε}

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Privacy: LDP with the **same** ε **Utility:** **Different errors** ?

Q: What is the **optimal** LDP mechanism?
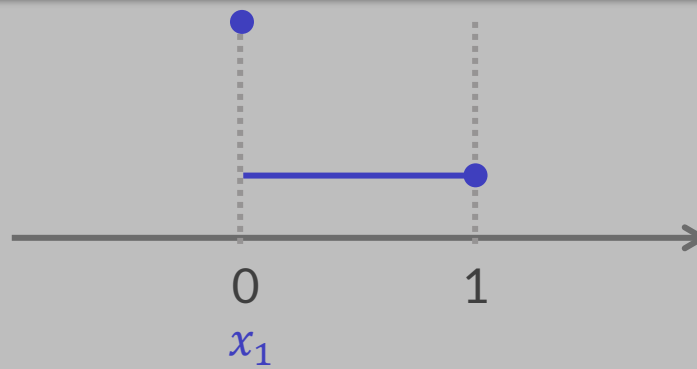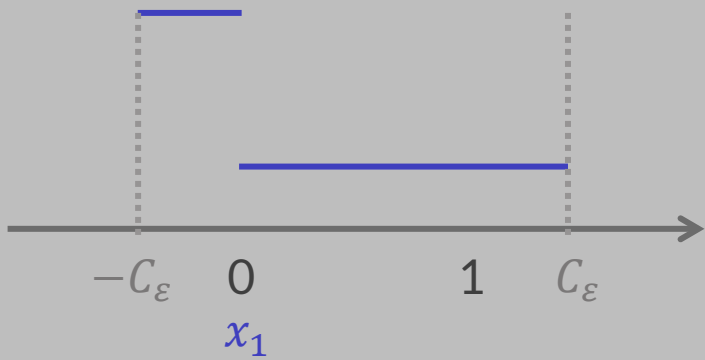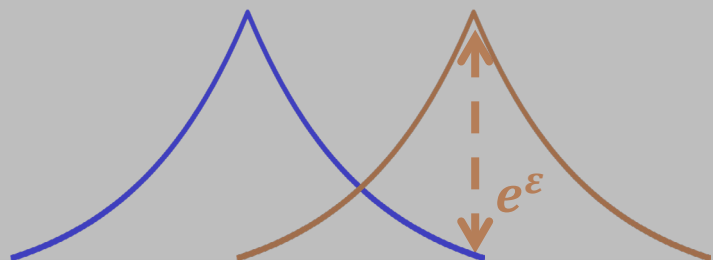
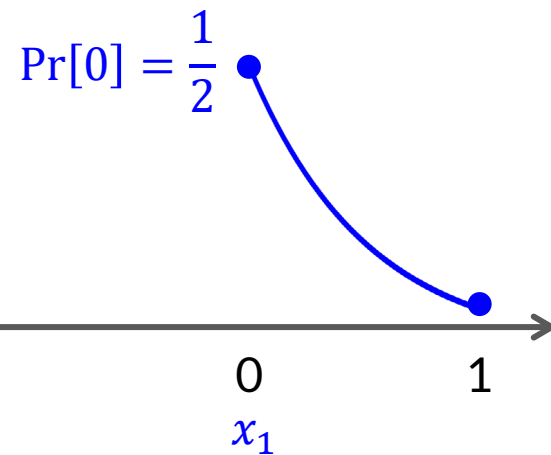$-C_{\varepsilon}$ 0   1 C_{ε}
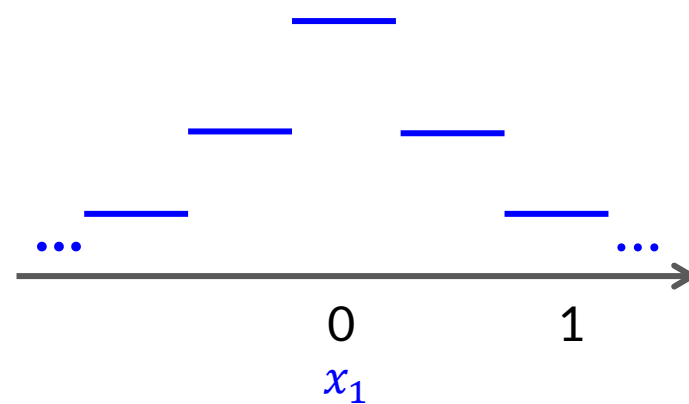
x_1

$\cdots$

0   1

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...   ...

Privacy: LDP with the **same** ε **Utility:** **Different errors**

Q: What is the **optimal piecewise-based** mechanism?

?

$-C_{\varepsilon}$ 0   1 C_{ε}

x_1

0   1

x_1

...
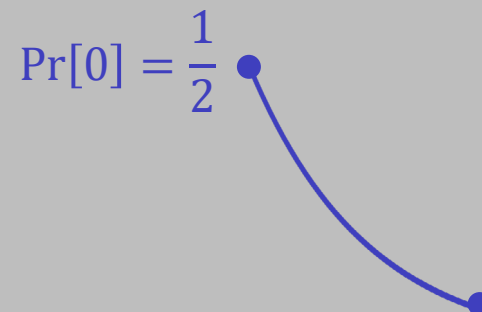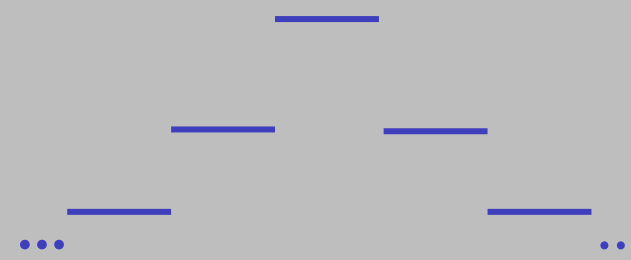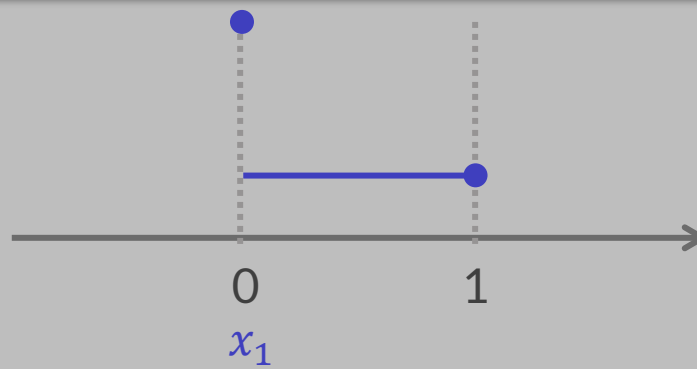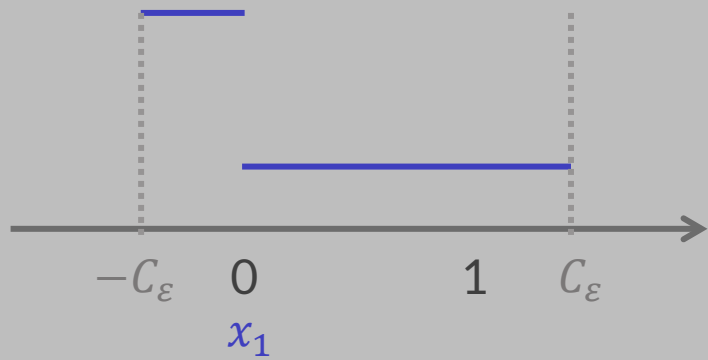
3-Piecewise Mechanism

- 3-piecewise distributions on bounded numerical domain $\mathcal{D} \to \widetilde{\mathcal{D}}$

  - given input x, sample output y from a distribution

$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_\varepsilon & \text{if } y \in [l_{x,\varepsilon}, r_{x,\varepsilon}] \\ \dfrac{p_\varepsilon}{\exp(\varepsilon)} & \text{if } y \in \widetilde{\mathcal{D}} \backslash [l_{x,\varepsilon}, r_{x,\varepsilon}] \end{cases}$$

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Sampling interval depends on x and ε

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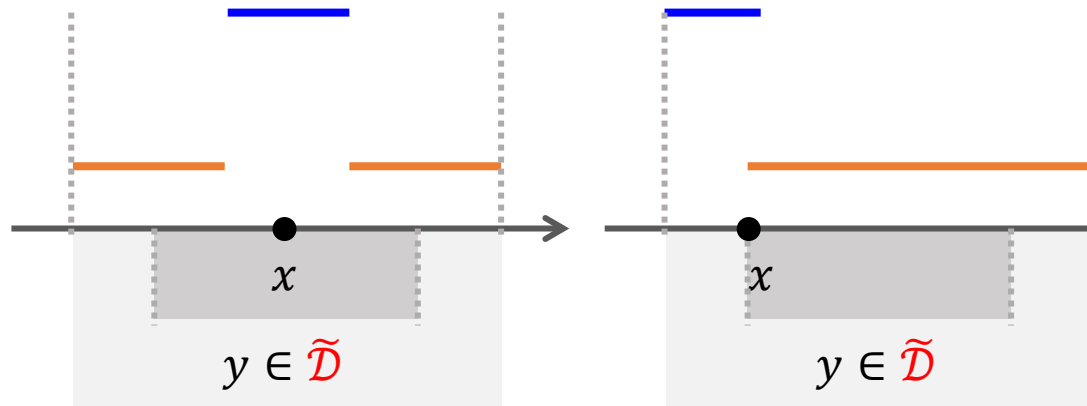
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⚠️ **NOT enough to study optimality** of piecewise-based mechanism

- only 3 pieces, two probabilities

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- **piecewise distribution can have more pieces and probabilities**

Larger ε

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Generalized Piecewise-based Mechanism

- **Most generalized version:** m-piecewise distribution

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- Error (data utility):

$$\mathcal{L}(y, x)$$

$$\mathcal{L}(y, x) := |y - x|^p$$

Generalized Piecewise-based Mechanism
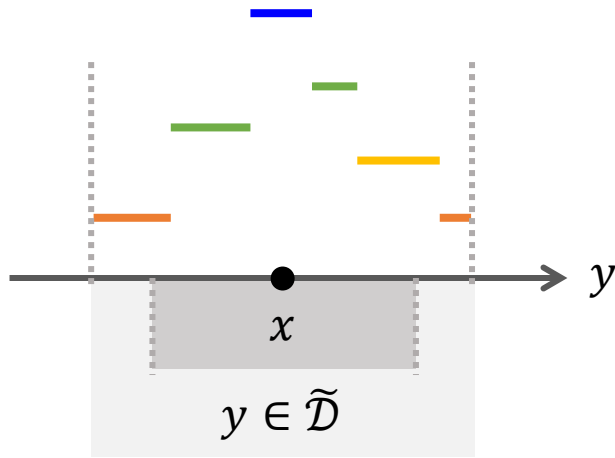
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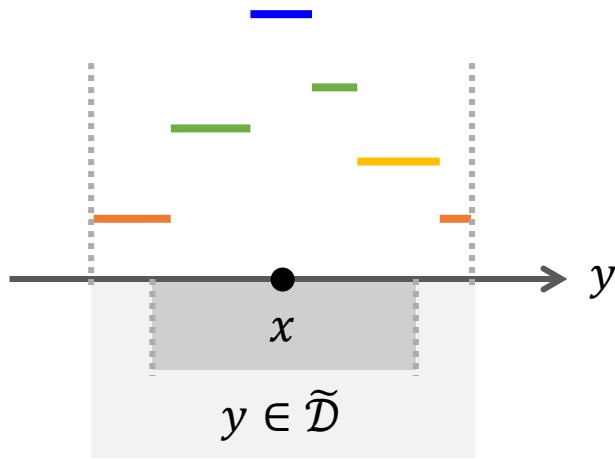
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- Expected error:

$$\int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y]\mathrm{d}y$$

Optimal Piecewise-based Mechanism

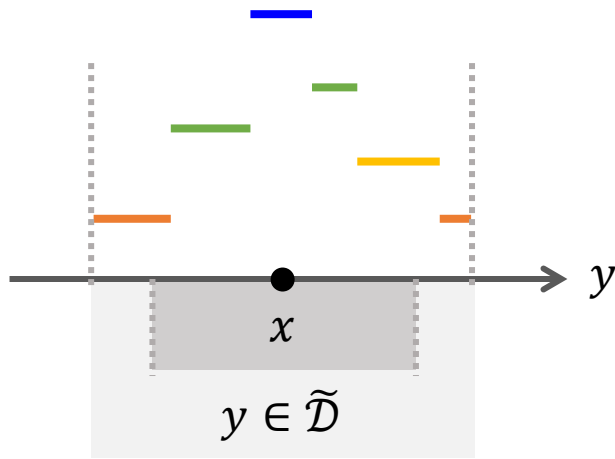- **Most generalized version:** m-piecewise distribution

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Find $\mathcal{M}$ to minimize the error at x

Optimal Piecewise-based Mechanism
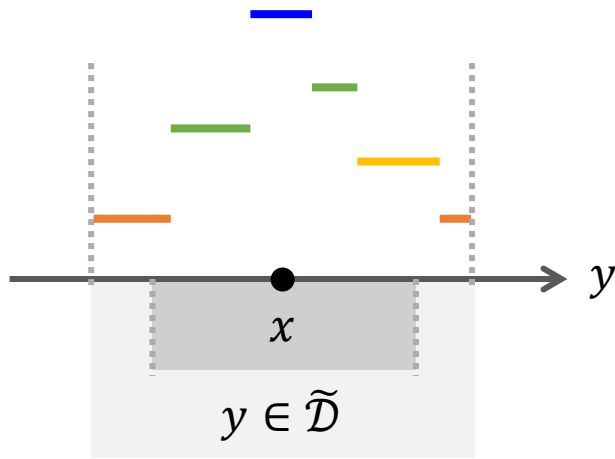
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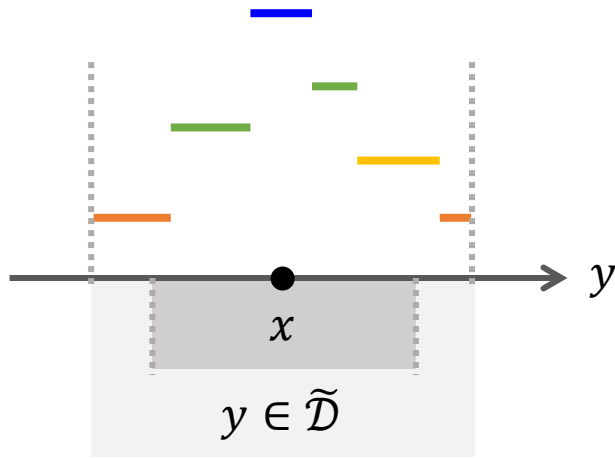
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Solved $\mathcal{M}$ is
the optimal piecewise-based mechanism

Mathematically $\equiv$ to find the **optimal piecewise distribution under the LDP constraint**

$$\min_{\mathcal{M}:\, p_i, l_i, r_i} \max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y]\mathrm{d}y$$

Find $\mathcal{M}$ to minimize the worst-case error

Challenges & Proofs

- Challenges

  1. min-max problem & multiple variables

  2. optimal results only for a specific m

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Inner max
has a closed form

Worst-case from
$x =$ endpoints of $\mathcal{D}$

Reduced to:

minimization problem

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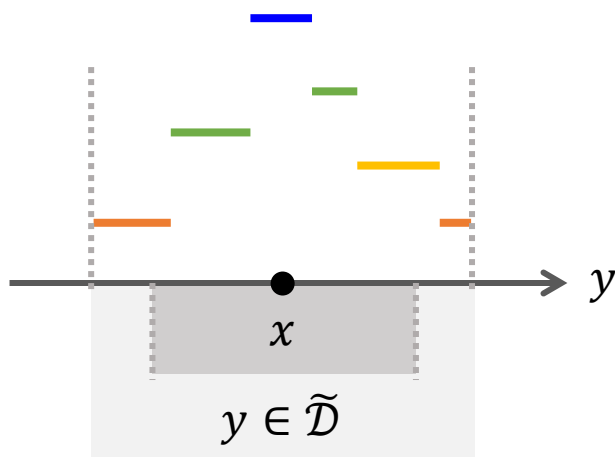
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2
The optimal #piece
is m

If optimal m-piecewise =
$(m + 1)$-piecewise
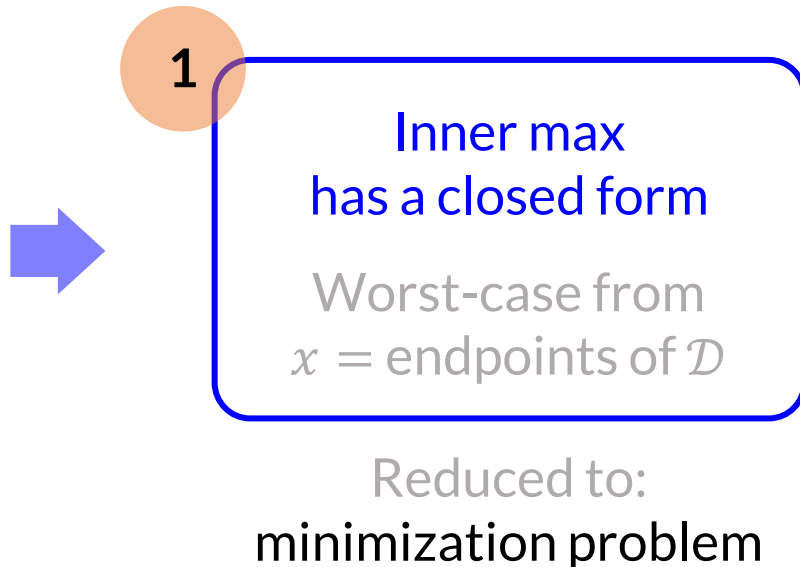
Optimal m:
Check $m = 1,2, \dots$ until optimal
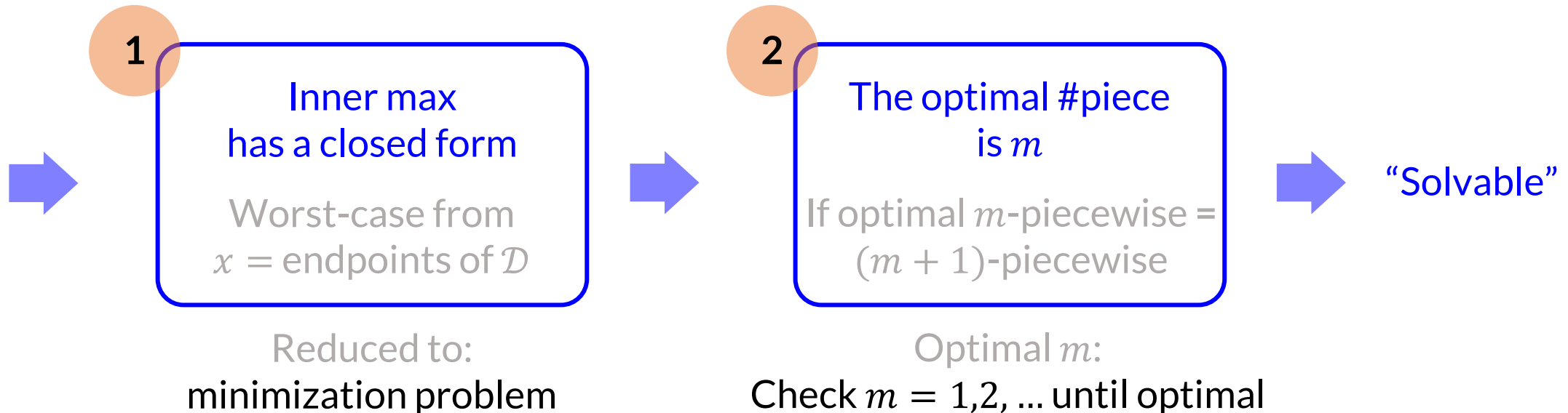
"Solvable"

- **When $m \geq 4$: Too many variables & non-linear**

$$\max_{x \in \{a,b\}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y, x) \, \mathrm{d}y$$

NOT Manually "Solvable" When $m \geq 4$

- **When $m \geq 4$: Too many variables & non-linear**

- Efficiently solved by off-the-shelf solvers, e.g. Gurobi

  - limitation: needs given ε

  - limitation: cannot provide closed-form $\mathcal{M}$: p_i, l_i, r_i (only optimal values)

  - can be used to analyze optimality

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Monte Carlo testing:
Optimality under 10^4 random ε

Hypothesis. For any domain $\mathcal{D} \to \mathcal{D}$, under error metrics $\mathcal{L}(y, x) \coloneqq |y - x|$ and $\mathcal{L}(y, x) \coloneqq (y - x)^2$, the optimal piecewise-based mechanism falls into 3-piecewise mechanism.

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different from existing instantiations
(closed-form $\mathcal{M}$ can be manually solved)

Closed Form Mechanism

- Optimal $\mathcal{M} : [0,1) \rightarrow [0,1)$ under $\mathcal{L} := |y - x|$

$$pdf[\mathcal{M}(x) = y] = \begin{cases} \exp\left(\dfrac{\varepsilon}{2}\right) & \text{if } y \in [l_{x,\varepsilon}, r_{x,\varepsilon}) \\[2ex] \exp\left(-\dfrac{\varepsilon}{2}\right) & \text{if } y \in [0,1) \backslash [l_{x,\varepsilon}, r_{x,\varepsilon}) \end{cases}$$

$$[l_{x,\varepsilon}, r_{x,\varepsilon}) = \begin{cases} [0, 2C) & \text{if } x \in [0, C) \\ x + [-C, C) & \text{if } x \in [C, 1 - C) \\ [1 - 2C, 1) & \text{otherwise} \end{cases} \qquad C = \frac{\exp(\varepsilon/2) - 1}{2(\exp(\varepsilon) - 1)}$$

- When $\varepsilon = 1$:
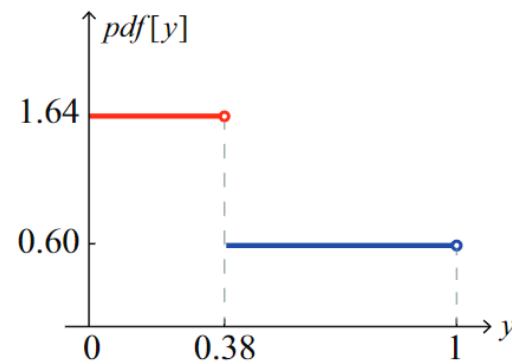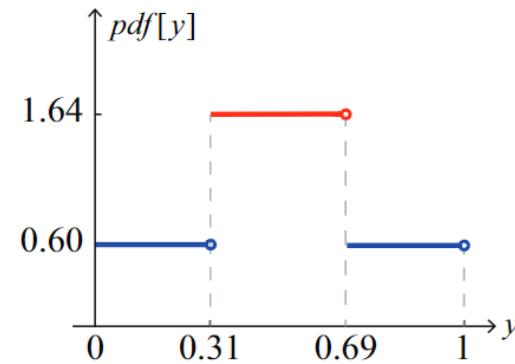
(a) $x = 0$. **(b)** $x = 0.5$.

Comparison of Expected Errors

Whole-domain error (i.e. each point in $\mathcal{D}$) **($\varepsilon = 2$)**

Worst-case error

Compressed PM, SW
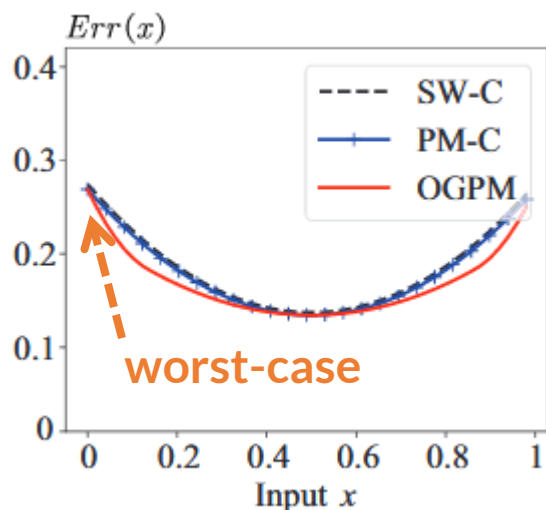
Original and **truncated** PM, SW
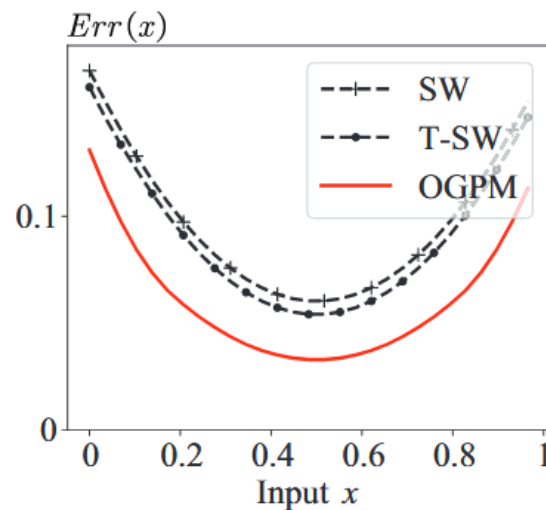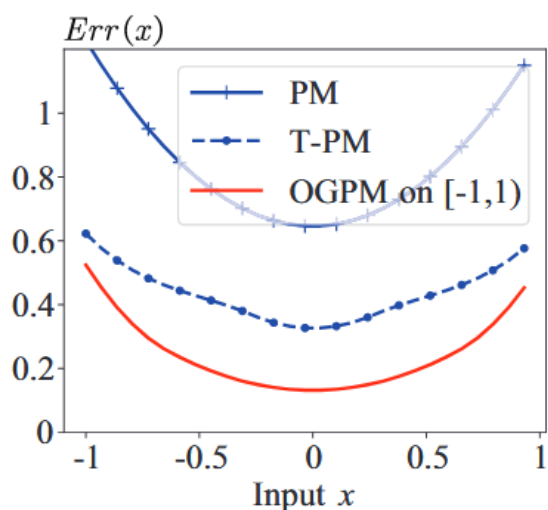
Non-piecewise-based

worst-case
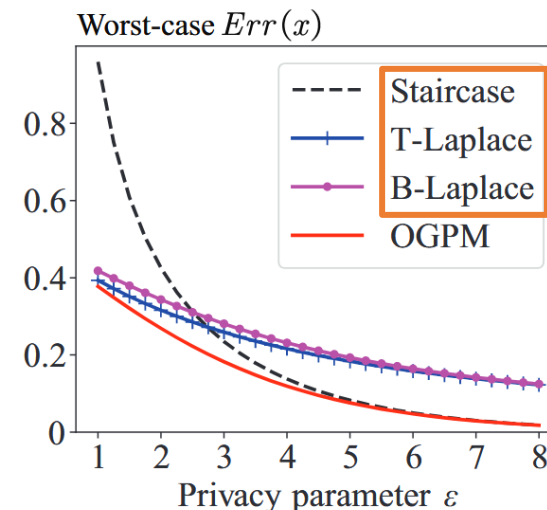
Lowest error

Lowest error

Circular Domain

- Different meaning of distance, e.g. $\text{distance}(0, 2\pi) = 0$
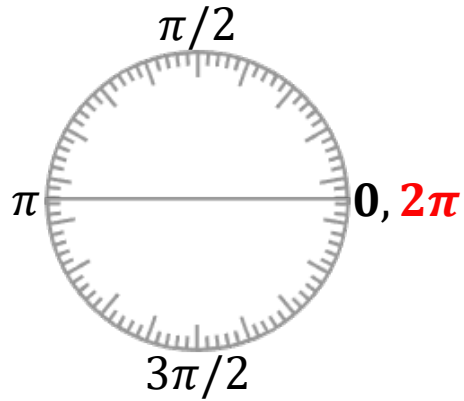
$\mathcal{L} \rightarrow \mathcal{L}_{\text{mod}}$

$$\mathcal{L}_{\text{mod}}(y, x) := \min(\mathcal{L}(y, x), \mathcal{L}(y, 2\pi - x))$$

Circular Domain

- Different meaning of distance, e.g. $\text{distance}(0, 2\pi) = 0$

$$\pi/2$$

$$\pi \quad\quad\quad 0, 2\pi$$

$$3\pi/2$$
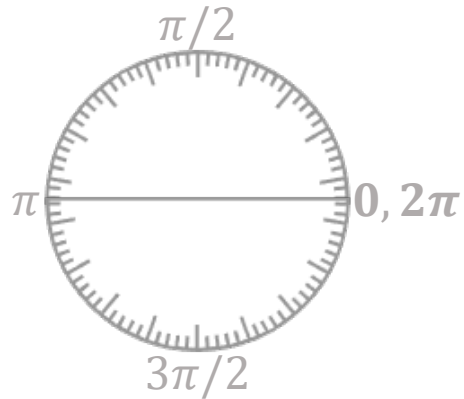
$$\mathcal{L} \to \mathcal{L}_{\text{mod}}$$

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$$\min_{\mathcal{M}:\, p_i, l_i, r_i} \max_{x \in [0, 2\pi]} \int_{\widetilde{\mathcal{D}}} \mathcal{L}_{\text{mod}}(y, x) \cdot pdf[\mathcal{M}(x) = y]\mathrm{d}y$$

Circular Domain

- Different meaning of distance, e.g. $\text{distance}(0, 2\pi) = 0$

$$\mathcal{L} \rightarrow \mathcal{L}_{\text{mod}}$$

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$$\min_{\mathcal{M}:\, p_i, l_i, r_i} \max_{x \in [0, 2\pi]} \int_{\widetilde{\mathcal{D}}} \mathcal{L}_{\text{mod}}(y, x) \cdot pdf[\mathcal{M}(x) = y] dy$$

- **Linking** to problems in the classical domain

min-max under $\mathcal{L}_{\text{mod}}$	=	min under $\mathcal{L}$ at π	min under $\mathcal{L}_{\text{mod}}$ at x	=	min under $\mathcal{L}$ at π + Transform

$$p_i \qquad\qquad l_{i,x}^{\text{mod}}, r_{i,x}^{\text{mod}}$$

Closed Form Mechanism

- Optimal $\mathcal{M} : [0, 2\pi) \to [0, 2\pi)$ under $\mathcal{L}_{\mathrm{mod}}$

$$pdf[\mathcal{M}(x) = y] = \begin{cases} \dfrac{1}{2\pi} \exp\left(\dfrac{\varepsilon}{2}\right) & \text{if } y \in [l_{x,\varepsilon}^{\mathrm{mod}}, r_{x,\varepsilon}^{\mathrm{mod}}) \\[2ex] \dfrac{1}{2\pi} \exp\left(-\dfrac{\varepsilon}{2}\right) & \text{if } y \in [0, 2\pi) \backslash [l_{x,\varepsilon}^{\mathrm{mod}}, r_{x,\varepsilon}^{\mathrm{mod}}) \end{cases}$$

$$[l_{x,\varepsilon}^{\mathrm{mod}}, r_{x,\varepsilon}^{\mathrm{mod}}) = [x - C, x + C) \mod 2\pi$$

$$C = \pi \frac{\exp(\varepsilon/2) - 1}{\exp(\varepsilon) - 1}$$

- When $\varepsilon = 1$:

$pdf[y] = 0.26$ $pdf[y] = 0.09$

(a) $x = 0$.   (b) $x = \pi/4$.
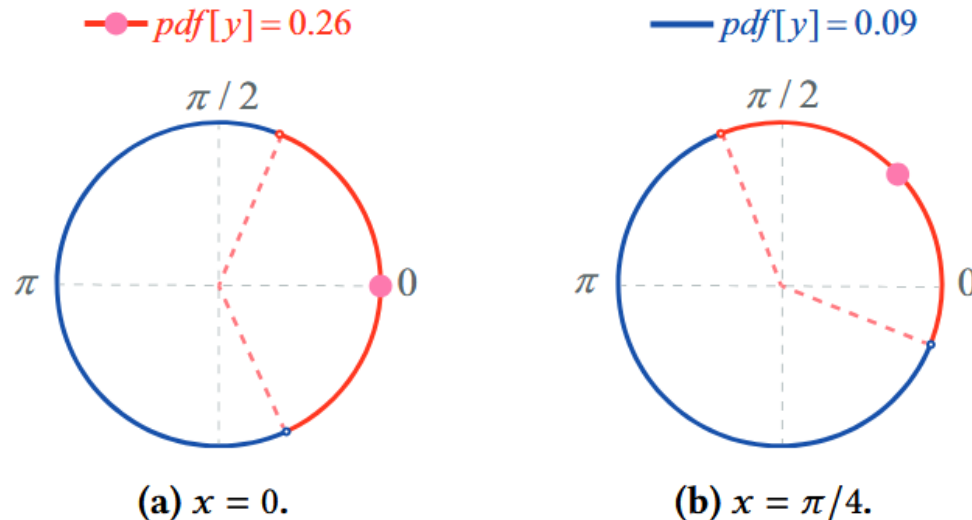
Comparison of Expected Errors

Whole-domain error ($\varepsilon = 2$)

PM-C, SW-C on the **flatten** domain
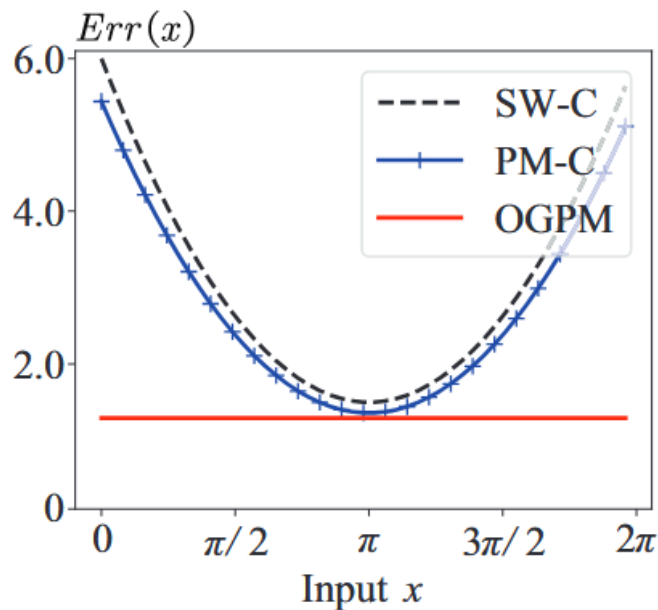
Worst-case error

Purkayastha mechanism [CCS'21]*
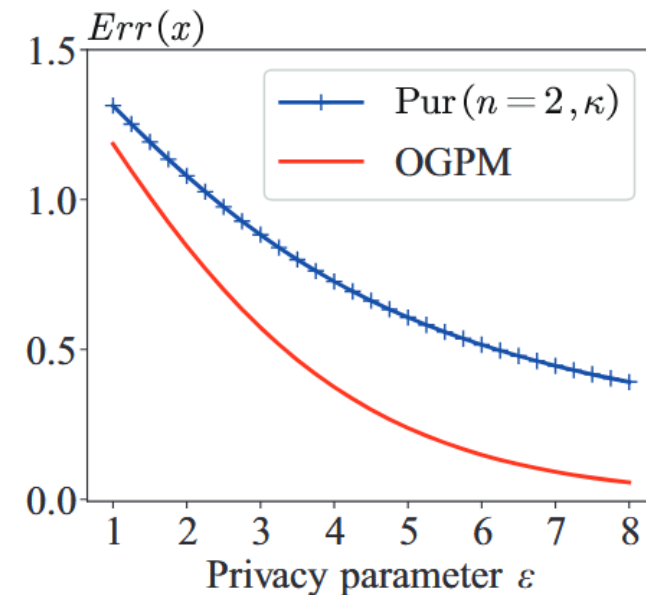
Lowest error

* Differential Privacy for Directional Data, *CCS'21*

Optimal Piecewise-based Mechanism for Collecting Bounded Numerical Data under LDP

LDP Mechanisms for Numerical Domain

Laplace mechanism: $[0,1] \to (-\infty, +\infty)$ Laplace + truncation: $[0,1] \to [0,1]$ Staircase: $[0,1] \to (-\infty, +\infty)$

$\Pr[0] = \frac{1}{2}$

e^ε

Privacy: LDP with the **same** ε **Utility:** **Different errors**   **?**

Q: What is the **optimal piecewise-based** mechanism?

$-C_\varepsilon$ 0 1 C_ε
x_1

0 1
x_1

3-Piecewise Mechanism

- 3-piecewise distributions on boun...

- given input x, sample output y fr...

⚠️ NOT enough to study optimality of piecewise-based mechanism
- only 3 pieces, two probabilities
- piecewise distribution can have more pieces and probabilities

Larger ε

x $y \in \tilde{\mathcal{D}}$ x $y \in \tilde{\mathcal{D}}$

$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_\varepsilon & \text{if } y \in [l_{x,\varepsilon}, r_{x,\varepsilon}] \\ \frac{p_\varepsilon}{\exp(\varepsilon)} & \text{if } y \in \tilde{\mathcal{D}} \setminus [l_{x,\varepsilon}, r_{x,\varepsilon}] \end{cases}$$

- Instantiations: PM [ICDE'19], SW [SIGMOD'20], PTT [TMC'24] (design different $p_\varepsilon, l_{x,\varepsilon}, r_{x,\varepsilon}$)

- different errors, but **without optimality**

Optimal Piecewise-based Mechanism

- **Most generalized version:** m-piecewise distribution

x $y \in \tilde{\mathcal{D}}$ y

$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_{1,\varepsilon} & \text{if } y \in [l_{1,x,\varepsilon}, r_{1,x,\varepsilon}] \\ p_{2,\varepsilon} & \text{if } y \in [l_{2,x,\varepsilon}, r_{2,x,\varepsilon}] \\ \dots \\ p_{m,\varepsilon} & \text{if } y \in [l_{m,x,\varepsilon}, r_{m,x,\varepsilon}] \end{cases}$$

$\frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \le e^\varepsilon$ (LDP constraint)

Solved $\mathcal{M}$ is the optimal piecewise-based mechanism

Mathematically ≡ to find the **optimal piecewise distribution** under the LDP constraint

$$\min_{\mathcal{M}: p_i, l_i, r_i} \max_{x \in \mathcal{D}} \int_{\tilde{\mathcal{D}}} \mathcal{L}(y,x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

Find $\mathcal{M}$ to minimize the worst-case error

Challenges & Proofs

- Challenges
  1. min-max problem & multiple variables
  2. optimal results only for a specific m

$$\min_{\mathcal{M}: p_i, l_i, r_i} \max_{x \in \mathcal{D}} \int_{\tilde{\mathcal{D}}} \mathcal{L}(y,x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

2. $i \in [m]$ 1. variables p_i, l_i, r_i

1 Inner max has a closed form
Worst-case from x = endpoints of $\mathcal{D}$
Reduced to: minimization problem

2 The optimal #piece is m
If optimal m-piecewise = $(m+1)$-piecewise
Optimal m: Check $m = 1,2, \dots$ until optimal

"Solvable"

Manually (Analytically) Solvable When $m = 3$

- When $m \ge 4$: Too many variables & non-linear
- Efficiently solved by off-the-shelf solvers, e.g. Gurobi
  - limitation: needs given ε
  - limitation: cannot provide closed-form $\mathcal{M}: p_i, l_i, r_i$
  - can be used to analyze optimality

$$\max_{x \in \{a,b\}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y,x) \mathrm{d}y$$

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different from existing instantiations (**closed-form** $\mathcal{M}$ can be manually solved)

Circular Domain

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$\pi/2$
π $0, 2\pi$
$3\pi/2$
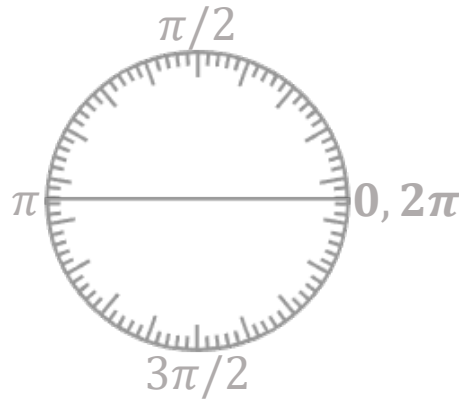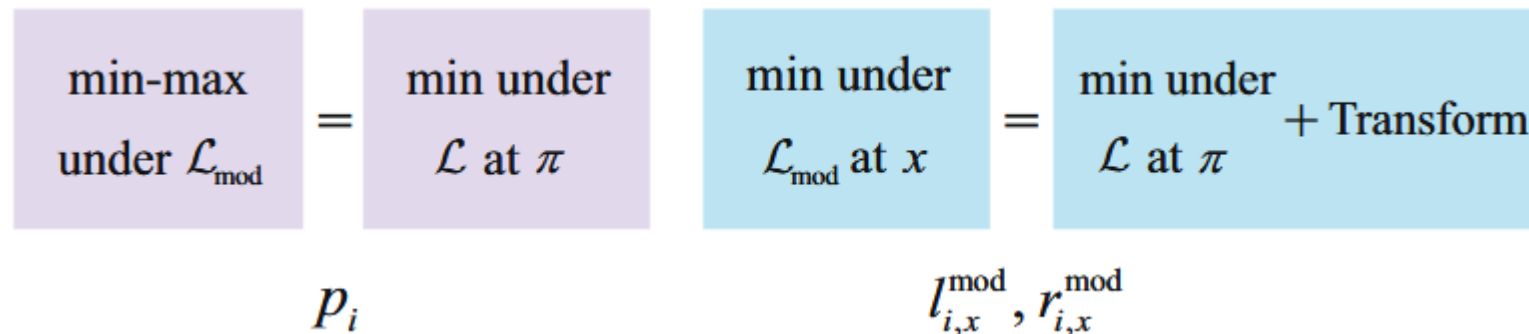
$\mathcal{L} \to \mathcal{L}_{\mathrm{mod}}$
$\mathcal{L}_{\mathrm{mod}}(y,x) := \min(\mathcal{L}(y,x), \mathcal{L}(y, 2\pi - x))$

$$\min_{\mathcal{M}: p_i, l_i, r_i} \max_{x \in [0, 2\pi]} \int_{\tilde{\mathcal{D}}} \mathcal{L}_{\mathrm{mod}}(y,x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

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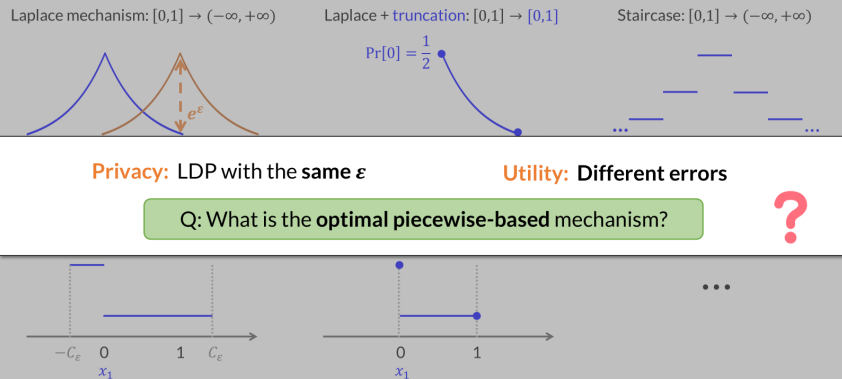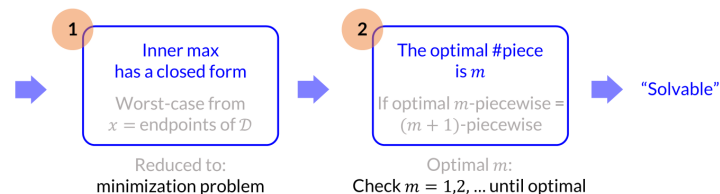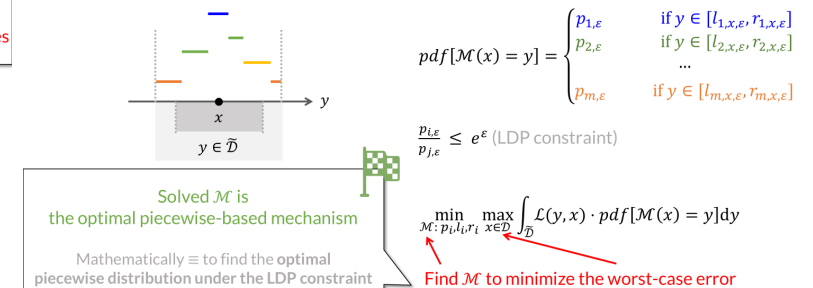| min-max under $\mathcal{L}_{\mathrm{mod}}$ | = | min under $\mathcal{L}$ at π | | min under $\mathcal{L}_{\mathrm{mod}}$ at x | = | min under $\mathcal{L}$ at π + Transform |

p_i $l_{i,x}^{\mathrm{mod}}, r_{i,x}^{\mathrm{mod}}$

Thank you!

Q&A

Optimality of LDP Mechanisms

- Optimal error (utility) under privacy level ε

  - many mechanisms are optimal in **order-of-magnitude**, e.g. $\Omega(\frac{1}{\sqrt{n}})$ for the counting query*

  - the staircase mechanism is optimal for **domain** $[0,1] \rightarrow (-\infty, +\infty)$[†]

  - the geometric mechanism is universally optimal if any **post-processing** is allowed, e.g. truncation[††]

* The Complexity of Differential Privacy, *book section of "Tutorials on the Foundations of Cryptography", 2017*

[†] The Staircase Mechanism in Differential Privacy, *journal version of ISIT'14*

[††] Universally Utility-maximizing Privacy Mechanisms, *STOC'09*

Optimality of LDP Mechanisms

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- **Specify the utility model** (conditions for optimality)

1 Error metric

$Err(\text{truth}, \text{rand})$

Err or $\mathbf{\Omega}(Err)$

2 Data domain & type of mechanisms

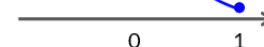Discrete / cont. $\mathcal{D} \to \widetilde{\mathcal{D}}$

Laplace-shape / piecewise

3 Post-processing

Laplace + truncation: $[0,1] \to [0,1]$

$\Pr[0] = \frac{1}{2}$

Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf\left[\mathcal{M}(x) = y\right] \mathrm{d}y = \max_{x \in \mathcal{D}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y, x) \, \mathrm{d}y \qquad (m\text{-piecewise distribution})$$

Optimal Piecewise-based Mechanism under LDP

Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y = \max_{x \in \mathcal{D}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y, x) \, \mathrm{d}y \qquad (m\text{-piecewise distribution})$$

convex function w.r.t x

Proof Intuitions

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$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y,x) \cdot pdf\left[\mathcal{M}(x) = y\right] \mathrm{d}y = \max_{x \in \mathcal{D}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y,x) \, \mathrm{d}y \qquad (m\text{-piecewise distribution})$$

$$= \max_{x \in \{a,b\}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y,x) \, \mathrm{d}y \qquad (\text{maximum principle})$$

Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = \underset{\cdot}{\overset{m}{\smash{\cdots}}} c^{r_i}$$

After merging redundant pieces

- Optimal #piece is m **if** optimal m-piecewise = (m+1)-piecewise

Error from an arbitrary piece

(≥ 0 variable)

if: $\quad \underset{e_1, e_2, e_3}{\min} \ e_1 + e_2 + e_3 = \underset{e_1, e_2, e_3, e_4}{\min} \ e_1 + e_2 + e_3 + e_4$

i.e. the error can't be lowered by arbitrary ≥ 0 variable

Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = \overset{m}{\cdots} \overset{c r_i}{\cdots}$$

> After merging redundant pieces

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i.e. the error can't be lowered by arbitrary ≥ 0 variable

then: $\displaystyle = \min_{e_1, e_2, e_3, e_4, e_5} e_1 + e_2 + e_3 + e_4 + e_5$

otherwise, $e_4 \leftarrow e_4 + e_5$ can further lower the error